ISRAEL JOURNAL OF MATHEMATICS, Vol. 35, No. 4, 1980

RIGID FACTORS OF ERGODIC TRANSFORMATIONS

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ABSTRACT

We prove a theorem concerning cartesian products of ergodic not necessarily measuring preserving transformations, using the notion of rigid factors for such transformations.

Let h be an ergodic (non-singular) transformation defined on separable probability space (S, Σ, μ). We mention that throughout this paper by ergodic we mean positively ergodic, i.e. we exclude the shift on the integers. Using a terminology introduced in [5] for m.p.t. we say that h has a (non-trivial) *rigid factor* if there exists a (non-trival) set A in Σ, satisfying

$$\mu(h^{-n_i}(A)\Delta A) \to 0, \qquad \text{as } n_i \to +\infty,$$

for some sequence of positive integers $n_i \to +\infty$. We note first that this is an invariance property of h, i.e. it is preserved under change of μ to an equivalent probability measure. We mention also the classification of invertible ergodic transformations into orbit types, e.g. as in [7]. In particular we have the infinite orbit types classifying the invertible ergodic transformations that do not accept a finite invariant measure, among which we distinguish the infinite invariant type (type II_∞) characterizing the ergodic transformations that accept an infinite σ-finite invariant measure. We prove the following:

THEOREM. *Let h be an ergodic transformation defined on a probability space (S, Σ, μ). Then the following are equivalent*:

(i) *h has a (non-trivial) rigid factor.*

(ii) *There exists an invertible ergodic transformation σ such that $h x \sigma$ is not ergodic.*

Received May 10, 1979 and in revised form November 13, 1979

(iii) *For every infinite orbit type there exists an invertible ergodic transformation* σ *of that type such that* $h \times \sigma$ *is not ergodic.*

(iv) *h has a (non-trivial) weakly rigid factor (i.e., a set A with* $1_h - n_{i_{(A)}} \to 1_A$ *weakly in* L_2*).*

Remarks. For the proof we will use the notion of eigenoperators introduced in [2]. In fact the statement (ii)→(i) was essentially obtained in [4] using the same technique but we will repeat some of the argument here for convenience, especially since an incorrect inference was made subsequently in [4] as is also indicated in [1], [5]. We mention also that, as has been pointed out by the referee, a statement similar to (ii)→(iv), along with many more results, can also be found in [1, theor. 6.1] in the more general setting of Markov processes. Therefore since also the direction (iii)→(ii) is trivial what is really new here is the statement (iv)→(iii), which can be seen as a generalization of the main result in [5] where h was assumed to be measure preserving. For the proof we need the following two basic results:

(A) If a homeomorphism in a compact metric space has an infinite recurrent orbit ($T^{n_i}z \to z$ for some sequence $n_i \to \infty$) then it accepts a (non-atomic) ergodic Borel probability measure [6].

(B) If a homeomorphism in a separable complete metric space accepts a (non-atomic) ergodic Borel probability measure then it also accepts such a measure from every infinite orbit type [7], [8].

Proof of the Theorem. Let h be an ergodic transformation defined on a probability space (S, Σ, μ). We note first that we can find always an equivalent probability measure, e.g. $\Sigma_0^{+\infty} \mu h^{-i}/2^{i+1}$, such that h induces in the (real or complex) Hilbert space $L_2(\Sigma)$ an injective continuous linear operator T, where $Tf(.) = f(h(.))$, a.e. We will assume w.l.o.g. that μ is in fact such a measure. We consider also the closed subset $C(\Sigma)$ of $L_2(\Sigma)$ given by the characteristic functions χ_A of sets A in Σ, and the corresponding restriction of $T: C(\Sigma) \to C(\Sigma)$ defined by $T\chi_A = \chi_{h^{-1}(A)}$. We have now that the metric on $C(\Sigma)$ inherited from $L_2(\Sigma)$ coincides with the usual metric $d(\chi_A, \chi_B) = \mu(A \Delta B)$ induced on $C(\Sigma)$ by identifying sets in Σ with their characteristic functions.

(ii)→(i). Let now σ be an invertible ergodic transformation defined on a probability space (S', Σ', μ') and $\chi(s, s')$ a non-trivial characteristic function that is invariant for $h \times \sigma$. Then the vector valued function $X(.) = \chi(s, .): S' \to L_2(\Sigma)$ is Borel measurable by [3, p. 196] and satisfies the eigenoperator equation $X(\sigma^{-1}(.)) = TH(.)$, μ'-a.e. We have also that the essential range of X is contained in $C(\Sigma)$ and does not consist of a single point. It follows that the measure $\mu' X^{-1}$ induced on $C(\Sigma)$ by X is a non-trivial ergodic Borel probability measure for the continuous map $T: C(\Sigma) \to C(\Sigma)$. However $C(\Sigma)$ being a separable complete metric space, it is well known (by the usual category argument) that the existence of such a measure implies the existence of a non-trival recurrent orbit for $T: C(\Sigma) \to C(\Sigma)$. It follows that for a non-trival set A in Σ we have $\mu(h^{-n_i}(A)\Delta A) \to 0$ for some sequence $n_i \to +\infty$ i.e. that h has a non-trivial rigid factor.

(i)$\to$(iii). We assume now that for a non-trivial set A in Σ we have $\mu(h^{-n_i}(A)\Delta A) \to 0$ for some sequence $n_i \to +\infty$ or equivalently if we set $z = \chi_A$ that the orbit $\{T^i z : i = 0, 1, 2, \cdots\} \subset L_2(\Sigma)$ is non-trivial recurrent. Considering first the case where this orbit is infinite and noting that it is also bounded we have that its closure $W(z)$ in the weak topology of $L_2(\Sigma)$ is compact, and $T: W(z) \to W(z)$ being injective and onto is in fact a homeomorphism. It follows from (A) that $T: W(z) \to W(z)$ accepts a non-atomic ergodic probability measure defined on the σ-algebra generated by the weak topology which, because $L_2(\Sigma)$ is a separable Hilbert space, coincides with the Borel σ-algebra, i.e. the σ-algebra generated by the norm tooplogy. It then follows from (B) that $T: W(z) \to W(z)$ accepts such measures from every infinite orbit type. If now m is such a measure, we consider the ergodic transformation $\sigma = T^{-1}$ defined on the probability space $(S' = W(z), B, m)$ where B denotes the Borel σ-algebra and we show that $h \times \sigma$ is not ergodic. Indeed the vector-valued function $X: S' \to L_2(\Sigma)$ defined by the inclusion map $W(z) \to L_2(\Sigma)$ is Borel measurable, essentially bounded, not a constant m-a.e., and satisfies the eigenoperator equation $X(\sigma^{-1}(.)) = TX(.)$, m-a.e. It follows by [3, p. 198] that the scalar-valued function $f(s, s') = X(s')$ is measurable. It is also not a constant $\mu \times m$-a.e. and invariant for $h \times \sigma$, which implies that $h \times \sigma$ is not ergodic, as claimed.

Finally we examine the case where the orbit $T^i z$, $i = 0, 1, 2, \cdots$, is finite. This assumption implies that T (equivalently h) has non-trivial eigenvalues that are (complex) roots of 1. Since the cartesian product of two ergodic transformations having a non-trivial eigenvalue in common is not ergodic it will suffice to show that for every infinite orbit type there exist ergodic transformations of that type having all the roots of 1 as eigenvalues. Indeed if we consider the minimal rotation on the monothetic compact metric group whose dual group is given by the roots of 1, we have that the Haar measure satisfies the assumption in (B) and hence the conclusion in (B) is also valid. The desired result follows then by the fact that the characters of the group are eigenfunctions having the required eigenvalues. Finally we note that the equivalence (i) $\Leftrightarrow$ (iv) is clear, [1].

Acknowledgements

This work was completed during a visit at the University of Warwick, in the winter of 1978–79. The author would like to thank especially Prof. W. Parry and Dr. K. Schmidt for their hospitality and assistance. Also the referee for pointing out [1] and for his other suggestions.

References

1. J. Aaronson, M. Lin and B. Weiss, *Mixing properties of Markov operators and ergodic transformations, and ergodicity of cartesian products*, Israel J. Math. **33** (1979), 198–224.

2. A. Beck, *Eigenoperators of ergodic transformations*, Trans. Amer. Math. Soc. **94** (1960), 118–129.

3. N. Dunford and I. T. Scwartz, *Linear Operators*, Part I, Interscience Publishers, 1966.

4. E. Flytzanis, *Ergodicity of the cartesian product*, Trans. Amer. Math. Soc. **186** (1973), 171–176.

5. H. Furstenberg and B. Weiss, *The Finite Multipliers of Infinite Ergodic Transformations*, Springer-Verlag L.N. 668,Proceedings, 1977.

6. Y. Katznelson and B. Weiss, *The construction of quasi-invariant measures*, Israel J. Math. **12** (1972), 1–4.

7. W. Krieger, *On Borel automorphisms and their quasi-invariant measures*, Math. Z. **151** (1976), 19–24.

8. K. Schmidt, *Infinite invariant measures on the circle*, Symposia Math. **21** (1977), 37–43.

9. P. Walters, *Some invariant σ-algebras for measure preserving transformations*, Trans. Amer. Math. Soc. **163** (1972), 357–368.

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